Mathematics for Physics: Additional Notes

June 24, 2024

Primed indices

p40, 130, 162, 232, 240, 254: When components of an intrinsic quantity change under a primed frame, the more accurate notation is e.g. $(v^\mu)'$ instead of v'^μ, since it is the components that are changing, not the vector itself. In contrast, under a change of frame the frame vectors themselves do change, making the more accurate notation e'_μ.

Morphisms

p3, 1.2 Defining mathematical structures and mappings: In category theory, morphisms have generalized definitions which can in some cases be distinct from the ones we give, which are common in abstract algebra.

Generalizing numbers

p6, 2.1 Generalizing numbers: The abelian group of positive reals under multiplication (isomorphic to the reals under addition via the logarithm) is often denoted $\mathbb{R}^+$ or $\mathbb{R}^\times$; both can be potential sources of confusion since the first might seem to imply the operation is addition, and the second does not explicitly signify that the elements must be positive (or alternatively non-zero).

Groups

p6-7, 2.1.1 Groups: Any permutation can be obtained from a composition of **transpositions**, or element exchanges; these transpositions are not unique, but the evenness or oddness (**parity**) of their number t is, and the **sign** of the permutation is defined as $(-1)^t$.

Rings

p8, 2.1.2 Rings: A ring without unity is called a rng ("ring without the i").

Modules

p9-10, 2.2 Generalizing vectors:

- To be explicit, for a right module, the given scalar multiplication rules are modified.
- Any unital ring R can itself be viewed as a R-module.

Vector spaces and complexification

p10, 2.2 Generalizing vectors:

- Bases generate a matrix for any vector space homomorphism (linear map), a particular one being a change of basis.
- The complexification of a real vector space can also be denoted $V^{\mathbb{C}}$.
- Decomplexification is also called realification.
- Given a $2n$-dimensional real vector space U, one can define a **complex structure** on U, defined to be a linear transformation $J \colon U \to U$ that squares to the negative identity; a basis for U is then $\{e_\mu, J(e_\mu)\}$, which determines an n-dimensional complex vector space W^J with basis e_μ.

Inner products

p11, 2.2.1 Inner products of vectors:

- Another term for anti-linear is conjugate-linear.
- An inner product as we have defined it on a complex vector space is also called a Hermitian inner product, and a complex inner product space is sometimes called a Hermitian inner product space, Hermitian space, or unitary space.
- Every finite-dimensional real or complex inner product space is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ with their standard inner products as defined, and $\langle v, v \rangle$ yields the same real value when applied the complex vector $v \in V$ or the decomplexified real vector in $V^{\mathbb{R}}$.
- The vectors orthogonal to a light-like vector in a Lorentzian signature are scalar multiples of itself and the space-like vectors orthogonal to its space-like component. This shows that the concept of an orthogonal complement (which together with the original subspace comprises the entire space) is not applicable to pseudo inner products.
- The definitions of pseudo inner product and signatures should have been put into this section, not the section on multilinear forms.
- The real quantity $\sqrt{\pm \langle v, v \rangle}$ is also sometimes called the "length" of v.
- The invariance of signature under a change of basis is known as **Sylvester's law of inertia**.
- The terms Euclidean and Minkowskian usually imply an inner product $\langle \hat{e}_\mu, \hat{e}_\nu \rangle = \eta_{\mu\nu}$ defined on the basis vectors of $\mathbb{R}^n$, which we denote $\mathbb{R}^{r,s}$ for an inner product of signature (r, s), a common but not universal convention.

Norms

p12, 2.2.2 Norms of vectors:

- The polarization identity can be written multiple ways:

$$\begin{aligned}\langle v, w \rangle &= \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) \\ &= \frac{1}{2}\left(\|v\|^2 + \|w\|^2 - \|v - w\|^2\right) \\ &= \frac{1}{2}\left(\|v + w\|^2 - \|v\|^2 - \|w\|^2\right).\end{aligned}$$

- Note that the parallelogram and polarization identities, which only involve the squared norm $\|v\|^2 = \langle v, v \rangle$, also hold for a pseudo inner product.

- Complex normed vector spaces are defined identically, but satisfy a different polarization identity

$$\langle v, w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\,\|v-iw\|^2 - i\,\|v+iw\|^2\right).$$

- Comparing the polarization identities immediately identifies the real part of the complex inner product as just the real inner product defined by the orthonormal basis in the decomplexification $V^{\mathbb{R}}$, i.e. $\mathrm{Re}\left(\langle v, w\rangle_{\mathbb{C}}\right) = \langle v, w\rangle_{\mathbb{R}}$. The imaginary part is then the inner product of vectors $\mathrm{Im}\left(\langle v, w\rangle_{\mathbb{C}}\right) = \langle v, -iw\rangle_{\mathbb{R}} = \langle iv, w\rangle_{\mathbb{R}}$ in $V^{\mathbb{R}}$. Note that both parts vanish for vectors in different complex planes, but for vectors in the same complex plane $\mathrm{Im}\left(\langle v, w\rangle\right)$ acts as a sort of "inverse" inner product on the vectors in $V^{\mathbb{R}}$: it vanishes if they are a real multiple of each other (parallel in $V^{\mathbb{R}}$), and is the squared norm for vectors related by an imaginary factor (orthogonal in the decomplexified complex plane), i.e. $\mathrm{Im}\left(\langle iv, v\rangle_{\mathbb{C}}\right) = \langle v, v\rangle_{\mathbb{R}}$. As we see later, $\mathrm{Im}\left(\langle v, w\rangle\right)$ is a symplectic form on $V^{\mathbb{R}}$.

- In a real inner product space, we can define the angle between vectors by $\cos\theta \equiv \langle v, w\rangle / (\|v\|\,\|w\|)$. If we then decompose v into components parallel and orthogonal to w, the parallel component is called the **orthogonal projection** of v onto w, and has length $v_{\|w} = \langle v, w\rangle / \|w\| = \|v\|\cos\theta$. In a complex vector space V, taking the real part of the cosine defines the **Euclidean angle** $\cos\theta_{\mathrm{E}} \equiv \mathrm{Re}\left(\langle v, w\rangle\right) / (\|v\|\,\|w\|)$, which is the angle between the vectors in the decomplexification $V^{\mathbb{R}}$; the orthogonal projection of v onto w in $V^{\mathbb{R}}$ is then $v_{\|w} = \mathrm{Re}\left(\langle v, w\rangle\right) / \|w\| = \|v\|\cos\theta_{\mathrm{E}}$.

- For two vectors in a single complex line in V, called a **holomorphic plane** in $V^{\mathbb{R}}$, we may write $\sin\theta_{\mathrm{E}} \equiv \mathrm{Im}\left(\langle z, u\rangle\right) / (\|z\|\,\|u\|)$, so that the component of z orthogonal to u is $z_{\perp u} = \mathrm{Im}\left(\langle z, u\rangle\right) / \|u\|$. In the more general case, two arbitrary complex vectors (assumed to be non-parallel in $V^{\mathbb{R}}$) have a **Kähler angle** defined by $\cos\theta_{\mathrm{K}}\sin\theta_{\mathrm{E}} \equiv \mathrm{Im}\left(\langle v, w\rangle\right) / (\|v\|\,\|w\|)$. $\cos\theta_{\mathrm{K}}$ is dependent only upon the plane in $V^{\mathbb{R}}$ defined by the two vectors, and is unity if this plane is holomorphic, while vanishing for vectors in orthogonal complex lines.

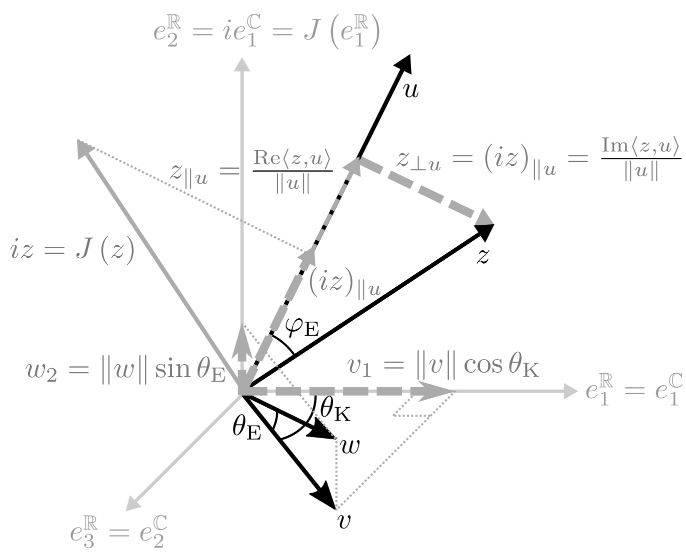

FIGURE 1: In the holomorphic plane defined by $e_1^{\mathbb{R}}$ and $e_2^{\mathbb{R}}$, the real part of the complex inner product $\langle z, u\rangle$ determines the parallel component $z_{\|u}$ (the orthogonal projection), while the imaginary part determines the orthogonal component $z_{\perp u}$; these can also be expressed in terms of the Euclidean angle as $z_{\|u} = \|z\|\cos\varphi_{\mathrm{E}}$ and $z_{\perp u} = \|z\|\sin\varphi_{\mathrm{E}}$. For two vectors v and w with an extra component in a direction orthogonal to the holomorphic plane, the Kähler angle is the angle between the vw plane and the holomorphic plane. We can see this in the figure by noting that since the $e_3^{\mathbb{R}}$ components of the vectors are parallel, the imaginary part of $\langle v, w\rangle$ does not include any contribution from this component. In the holomorphic plane, v has no imaginary component, so we have $\mathrm{Im}\langle v, w\rangle = v_1 w_2 = \|v\|\cos\theta_{\mathrm{K}}\,\|w\|\sin\theta_{\mathrm{E}}$, the defining relation for the Kähler angle. This geometric view of the Kähler angle as the angle between planes remains valid if the line of intersection between the planes is not along the imaginary axis, or if v has an imaginary component; but for vectors with components in multiple holomorphic planes, the situation is more complicated.

- Finally, taking the modulus of the cosine defines the **Hermitian angle** $\cos\theta_{\mathrm{H}} \equiv |\langle v, w\rangle| / (\|v\|\,\|w\|)$, where $v^{\mathbb{C}}_{\|w} = \|v\|\cos\theta_{\mathrm{H}}$ is the (complex) orthogonal projection of v onto w. The **pseudo-angle** is then defined by

$\langle v,w\rangle \equiv |\langle v,w\rangle|\, e^{i\theta_{\mathrm{P}}}$. Like the Kähler angle, the Hermitian angle vanishes for vectors in orthogonal complex lines; for two vectors in the same holomorphic plane in $V^{\mathbb{R}}$, the Hermitian angle is unity and the pseudo-angle is just the Euclidean angle.

△ It is important to remember that a Euclidean angle of $\pi/2$ does not ensure a vanishing complex inner product, and that parallel vectors in a complex V may be orthogonal using the corresponding real inner product in $V^{\mathbb{R}}$.

Symplectic forms

p15, 2.2.4 Orthogonality of vectors:

- A vector space along with a symplectic form is called a **symplectic vector space**. Every finite-dimensional symplectic vector space is isomorphic to $\mathbb{R}^{2n}$ under the standard symplectic form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

  (where $J_{\mu\nu} \equiv J\left(e_\mu, e_\nu\right)$ and I is the identity matrix), which is then isomorphic to $\mathbb{C}^n$ under the imaginary part of the complex inner product.

- The requirement of even dimension can be seen by considering nondegeneracy in light of the determinant of the matrix form.

Algebras

p16, 2.2.5 Algebras: multiplication of vectors: the universal enveloping algebra is infinite dimensional.

p17, 2.2.6 Division algebras:

- Note that the norm via the conjugate implies two sided inverses for all normed real division algebras, namely $v^{-1} = v^*/\left\|v\right\|^2$.
- It turns out that any subalgebra of $\mathbb{O}$ generated by two elements is associative.

Tensor product

p21, 2.3.3 The tensor product: The list of isomorphisms at the end of the section are as real algebras. Complexification is equivalent to tensoring with the complex numbers, i.e. $V^{\mathbb{C}} \cong V \otimes \mathbb{C}$, so the first isomorphism can be viewed as the complexification of $\mathbb{C}$ as a real algebra. An explicit isomorphism is $a(1\otimes 1) + b(i\otimes 1) + c(1\otimes i) + d(i\otimes i) \mapsto ((a+d)+i(b-c), (a-d)+i(b+c))$, or in the reverse direction $(z,w) \mapsto \frac{z}{2}(1\otimes 1 + i\otimes i) + \frac{w}{2}(1\otimes 1 - i\otimes i)$. Note that the original algebra is thus embedded as $a+ib \mapsto (a+ib, a+ib)$. We can then apply this isomorphism to each matrix element in $\mathbb{C}(n)$ as a real algebra to get $\mathbb{C}(n)^{\mathbb{C}} \cong \mathbb{C}(n)\otimes\mathbb{C} \cong \mathbb{C}(n)\oplus\mathbb{C}(n)$, where again uncomplexified elements are mapped as $v \mapsto (v,v)$.

Vector algebras

p27, Chapter 3 Vector algebras: Although noted in Section 3.1, the assumption that vector spaces are finite dimensional and real applies to the entire chapter.

Combinatorial notations

p30-31, 3.1.3 Combinatorial notations:

- The permutation symbol is defined in terms of parity to be +1 for even index permutations, −1 for odd, and 0 otherwise.
- The **generalized Kronecker delta**

$$\delta^{\nu_1\cdots\nu_k}_{\mu_1\cdots\mu_k} \equiv \sum_{\pi} \operatorname{sign}(\pi)\, \delta^{\nu_{\pi(1)}}_{\mu_1} \cdots \delta^{\nu_{\pi(k)}}_{\mu_k}$$

  gives the sign of the permutation of upper versus lower indices and vanishes if they are not permutations or have a repeated index. We can then relate this to the permutation symbol:

$$\begin{aligned} \delta^{\nu_1\cdots\nu_k}_{\mu_1\cdots\mu_k} &= \frac{1}{(n-k)!}\varepsilon^{\nu_1\cdots\nu_k\lambda_{k+1}\ldots\lambda_n}\varepsilon_{\mu_1\cdots\mu_k\lambda_{k+1}\ldots\lambda_n} \\ \Rightarrow \varepsilon^{\lambda_1\cdots\lambda_n}\varepsilon_{\lambda_1\ldots\lambda_n} &= n! \end{aligned}$$

- The norm of the exterior product of vectors $v_\mu = M^\nu{}_\mu \hat{e}_\nu$ is the absolute value of the determinant of the matrix, which equals the volume of the parallelepiped defined by the vectors if the basis is orthonormal.
- Since the specific vectors in $P = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ can have many values without changing P itself (e.g. $v \wedge w = (v + w) \wedge w$), a more accurate visualization might be the oriented subspace associated with the parallelepiped along with a basis-independent specification of volume. In particular, the change of basis formula above means that given any pseudo inner product, P can always be expressed as the exterior product of k orthogonal vectors.

Hodge star

p33, 3.1.4 The Hodge star: The first print edition of the book uses the definition $A \wedge C = \langle *A, C\rangle\, \Omega$ for $A \in \Lambda^k V$, $C \in \Lambda^{n-k}V$, which prefixes our current Hodge star by the factor $(-1)^s$. To be consistent with most physics texts our current Hodge star is defined such that for any $A, B \in \Lambda^k V$ we have

$$A \wedge *B = \langle A, B\rangle\, \Omega.$$

From this we immediately obtain

$$A \wedge *A = \langle A, A\rangle\, \Omega.$$

For $\hat{A} \equiv \hat{e}_1 \wedge \cdots \wedge \hat{e}_k$ and $\hat{C} \equiv \hat{e}_{k+1} \wedge \cdots \wedge \hat{e}_n$, we then have $*\hat{A} = \left\langle \hat{A}, \hat{A} \right\rangle \hat{C}$. For n-dimensional V with unit n-vector Ω and pseudo inner product of signature (r, s) we have:

- $*\Omega = (-1)^s \Rightarrow (*C)\, \Omega = (-1)^s\, C$ if $C \in \Lambda^n V$
- $*1 = \Omega \Rightarrow \langle *a, \Omega\rangle = (-1)^s\, a$ if $a \in \Lambda^0 V$
- $**A = (-1)^{k(n-k)+s}\, A = (-1)^{k(n-1)+s}\, A$, where $A \in \Lambda^k V$
- $A \wedge *B = B \wedge *A$ if $A, B \in \Lambda^k V$
- $*(A \wedge *B) = \langle A \wedge *B, \Omega\rangle = (-1)^s \langle A, B\rangle$ if $A, B \in \Lambda^k V$

Graded algebras

p34, 3.1.5 Graded algebras: For clarity we should explicitly say that we will assume gradation weights take integer values.

Clifford algebra Hodge star

p35, 3.1.6 Clifford algebras: With the new Hodge star definition we have

$$*A = (-1)^{\frac{k(k-1)}{2}} A\Omega,$$

and the following updated table:

	(2, 0)	(3, 0)	(3, 1)	(1, 3)
$*v$	$v\Omega = -\Omega v$	$v\Omega = \Omega v$	$v\Omega = -\Omega v$	$v\Omega = -\Omega v$
$*B$	$-B\Omega = -\Omega B$	$-B\Omega = -\Omega B$	$-B\Omega = -\Omega B$	$-B\Omega = -\Omega B$
$*T$		$-T\Omega = -\Omega T$	$-T\Omega = \Omega T$	$-T\Omega = \Omega T$

Table 0.1: The Hodge dual in terms of Clifford products in common signatures.
Notes: here we have $v \in V$, $B \in \Lambda^2 V$, and $T \in \Lambda^3 V$.

Geometric algebra Hodge star

p37, 3.1.7 Geometric algebra: With the new Hodge star definition we have $*A = \widetilde{A}\Omega$.

Musical isomorphisms

p39, 3.2.1 The structure of the dual space:

- It is important to remember that when the inner product is not positive definite, the signs of components may change under these isomorphisms.
- If the components are in terms of an arbitrary (non-orthonormal) basis, then as we see in a subsequent section, the components change their values as well, since $\eta_{\lambda\mu}$ is replaced by the metric tensor in the above analysis.
- The view of a 1-form as a projection requires a positive definite inner product.
- The explanation of the depiction of 1-forms as "receptacles" is better placed here than in Section 6.3.3, The Lie derivative of an exterior form.
- When depicting φ as changing linearly, the length L of the 1-form representation changes more generally like $L \mapsto L/(1 + r\varepsilon)$ for some scaling factor r, while a linearly changing vector representation would change like $L \mapsto L(1 + r\varepsilon)$.

Another common graphical device is to represent φ as a density of "surfaces" where the value of $\varphi(v)$ is the number of surfaces "pierced" by the arrow. Figure 2 covers some non-intuitive aspects of these visualizations.

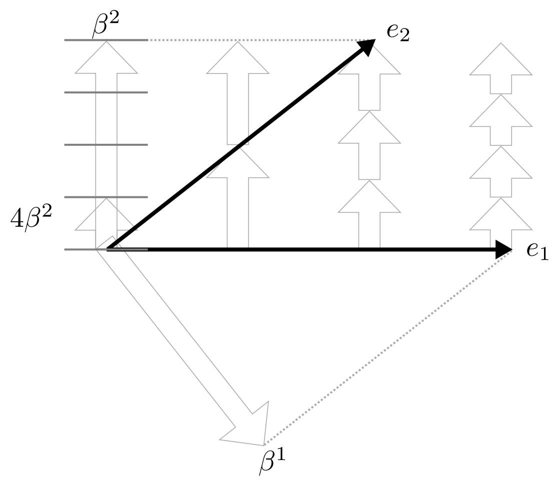

FIGURE 2: Depicting a 1-form φ as the associated vector $\varphi^{\Uparrow}$ or as a density of surfaces has consequences that can be non-intuitive. When orthogonality corresponds to right angles in a figure, an orthonormal basis and its dual basis appear as identical arrows; in the figure, we see that for a non-orthonormal basis, the dual basis does not appear to either be parallel to the basis or to have identical lengths. We also see that quadrupling the value of the 1-form means quartering its length in the figure, or equivalently quadrupling the density of surfaces pierced by arrows. This means that when depicting a linearly changing 1-form as above, the length L of the associated vector changes like $L \mapsto L/(1 + r\varepsilon)$ for some scaling factor r, which doesn't appear linear as a vector representation would, whose length changes like $L \mapsto L(1 + r\varepsilon)$.

Tensors

p40, 3.2.2 Tensors:

- A tensor of type $(k, 0)$ is called a **contravariant tensor**, with **covariant tensors** being of type $(0, k)$, and other tensor types being called **mixed tensors**.
- The meanings of tensor rank and order are often swapped. Another potential source of confusion is that a mixed tensor is not the opposite of a pure tensor.

Abstract index notation

p42, 3.2.4 Abstract index notation:

- Taking the tensor direct product of two tensors and then contracting all opposite indices is also called the contraction of the two tensors, i.e. the contraction of $S^{ab}{}_c$ and T_{def} is $C_{cf} = S^{ab}{}_c T_{abf}$.
- g^{ab} is called the **dual metric tensor** (AKA conjugate metric tensor), and the more directly derived relationship showing the consistency of index raising and lowering is $g^{ab}g_{ac}g_{bd} = g_{cd}$, not $g^{ab} = g^{ac}g^{bd}g_{cd}$. $g^{ab}g_{ab}$ is equal to the dimension of V.
- The contraction of any two symmetric indices with any two anti-symmetric indices vanishes, e.g. if the (first) second tensor is (anti) symmetric in the first two indices then

$$S^{abc}T_{abd} = -S^{bac}T_{bad} = -S^{abc}T_{abd},$$

where in the last step we relabel "dummy" indices summed over. Similarly, any tensor with overlapping anti-symmetric and symmetric indices vanishes, e.g. if the (first) second two indices are (anti) symmetric then

$$T^{abc} = -T^{bac} = -T^{bca} = T^{cba} = T^{cab} = -T^{acb} = -T^{abc}.$$

- Only tensors of order 2 are the sum of symmetrized and anti-symmetrized tensors.

Tensors as multi-dimensional arrays

p43, 3.2.5 Tensors as multi-dimensional arrays: Note that the initial expression means that in terms of the tensor as a multilinear mapping we have

$$T^{\mu_1 \cdots \mu_m}{}_{\lambda_1 \ldots \lambda_n} = T\left(\beta^{\mu_1}, \cdots, \beta^{\mu_m}, e_{\lambda_1}, \cdots, e_{\lambda_n}\right).$$

△ It is important to remember that a tensor $T^{\mu\nu}$ or $T_{\mu\nu}$ can be written as a matrix of scalars, but linear algebra operations only are valid for linear operators $T^{\mu}{}_{\nu}$. A similar source of potential confusion is that the (anti-)symmetry of $T^{\mu\nu}$ or $T_{\mu\nu}$ is basis independent, while that of $T^{\mu}{}_{\nu}$ is not.

△ It is important to remember that index lowering/raising does not apply to basis vectors, e.g. $\beta^{\mu} \overset{\text{no}}{=} g^{\mu\nu} e_{\nu}$ makes no sense since we cannot equate a 1-form to a sum of vectors.

As a component matrix, the metric tensor satisfies $g^{\mu}{}_{\lambda} g^{\lambda}{}_{\nu} = g^{\mu\lambda} g_{\lambda\nu} = g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} = I$, hence the dual metric tensor g^{ab} is also called the **inverse metric tensor**. Recalling the transformation of the top exterior product of basis vectors from the section on combinatorial notations, we can derive an expression for the unit n-vector in an arbitrary basis $e_{\mu} = M^{\nu}{}_{\mu} \hat{e}_{\nu}$ of the same orientation by using the component array of a metric of signature (r, s) in that basis:

$$\begin{aligned}
g_{\mu\nu} &= g\left(e_{\mu}, e_{\nu}\right) \\
&= g\left(M^{\lambda}{}_{\mu} \hat{e}_{\lambda}, M^{\sigma}{}_{\nu} \hat{e}_{\sigma}\right) \\
&= \sum_{\lambda} M^{\lambda}{}_{\mu} M^{\lambda}{}_{\nu} g\left(\hat{e}_{\lambda}, \hat{e}_{\lambda}\right) \\
&= \left(M^{T} \tilde{M}\right)_{\mu\nu} \\
\Rightarrow \det(g) &= \det\left(M^{T} \tilde{M}\right) \\
&= \pm \left(\det(M)\right)^{2} \\
\Rightarrow e_1 \wedge \cdots \wedge e_n &= \sqrt{|\det(g)|}\, \hat{e}_1 \wedge \cdots \wedge \hat{e}_n \\
\Rightarrow \hat{\beta}^1 \wedge \cdots \wedge \hat{\beta}^n &= \sqrt{|\det(g)|}\, \beta^1 \wedge \cdots \wedge \beta^n
\end{aligned}$$

Here $\tilde{M}$ is M with negative entries for every row λ where $g\left(\hat{e}_{\lambda}, \hat{e}_{\lambda}\right) = -1$, whose determinant is thus changed by a sign when s is odd.

△ It is important to remember that the element $g^{\mu\nu}$ is the entry in row μ and column ν of the inverse of the component matrix $g_{\mu\nu}$; in particular, $g^{\mu\nu} g_{\mu\nu} = r + s \neq 1$.

△ It is important to remember that the number $\det(g)$ is the determinant of the matrix with element $g_{\mu\nu}$ in row μ and column ν, and that it depends on both the basis and the inner product.

△ The symbol g is frequently used to denote $\det(g)$, and sometimes $\sqrt{|\det(g)|}$, in addition to denoting the metric tensor itself.

Exterior forms as completely anti-symmetric tensors

p45, 3.3.2 Exterior forms as completely anti-symmetric tensors: Also note that this isomorphism between the exterior product and the tensor product can be similarly used to identify the exterior product of vectors with a completely anti-symmetric contravariant tensor. In the following section we identify exterior forms with lower index anti-symmetric arrays; we can similarly identify the exterior product of vectors with upper index anti-symmetric arrays.

Exterior forms as anti-symmetric arrays

p46, 3.3.3 Exterior forms as anti-symmetric arrays: The equality of form and tensor arrays under our conventions is more clearly written without recourse to the shared linear mapping view, i.e.

$$\varphi \mapsto \frac{1}{k!}\varphi_{\mu_1\ldots\mu_k}\sum_{\pi}\operatorname{sign}(\pi)\bigotimes_{i}\beta^{\pi(i)} = \varphi_{\mu_1\ldots\mu_k}\beta^{\mu_1}\otimes\cdots\otimes\beta^{\mu_k}$$

This means that as with tensors, in terms of the k-form as a multilinear mapping we have

$$\varphi_{\mu_1\ldots\mu_k} = \varphi\left(e_{\mu_1},\cdots,e_{\mu_k}\right).$$

In particular, for an n-form we have

$$\begin{aligned}\varphi &= \varphi_{1\ldots n}\beta^1\wedge\cdots\wedge\beta^n\\ &= \varphi_{\mu_1\ldots\mu_n}\beta^{\mu_1}\otimes\cdots\otimes\beta^{\mu_n}\\ \Rightarrow \varphi\left(e_{\mu_1},\cdots,e_{\mu_n}\right) &= \varphi_{\mu_1\ldots\mu_n}.\end{aligned}$$

Since exterior forms are built from only the dual space V^*, in this context we will also use the symbol Ω to refer to the unit n-form. In an arbitrary basis we can then write

$$\begin{aligned}\Omega &= \sqrt{|\det(g)|}\,\beta^1\wedge\cdots\wedge\beta^n\\ &= \sqrt{|\det(g)|}\,\varepsilon_{1\ldots n}\beta^1\wedge\cdots\wedge\beta^n\\ &= \sqrt{|\det(g)|}\,\varepsilon_{\mu_1\ldots\mu_n}\beta^{\mu_1}\otimes\cdots\otimes\beta^{\mu_n},\end{aligned}$$

where $\epsilon_{\mu_1\ldots\mu_n}\equiv\sqrt{|\det(g)|}\,\varepsilon_{\mu_1\ldots\mu_n}$ is therefore the array of a tensor, sometimes called the **Levi-Civita tensor**. The component array expression for the exterior product of a j-form φ and a k-form ψ is then

$$(\varphi\wedge\psi)_{\mu_1\cdots\mu_{j+k}} = \frac{1}{j!k!}\varphi_{\nu_1\cdots\nu_j}\psi_{\nu_{j+1}\cdots\nu_{j+k}}\delta^{\nu_1\cdots\nu_{j+k}}_{\mu_1\cdots\mu_{j+k}}.$$

In particular, for two 1-forms we have

$$(\varphi\wedge\psi)_{\mu\nu} = \varphi_\mu\psi_\nu - \varphi_\nu\psi_\mu.$$

△ A potential source of confusion is that using abstract index notation one may write $\varphi_a\wedge\psi_b$, but $(\varphi_a\wedge\psi_b)\,v^a w^b \neq \varphi_a v^a\wedge\psi_b w^b = \varphi_a v^a\psi_b w^b$.

The component expression for the inner product of two k-forms is

$$\langle\varphi,\psi\rangle_{\text{form}} = \frac{1}{k!}\varphi_{\mu_1\ldots\mu_k}\psi^{\mu_1\ldots\mu_k},$$

and that of the Hodge star of a k-form is

$$\begin{aligned}(*\varphi)_{\mu_{k+1}\ldots\mu_n} &= \frac{\sqrt{|\det(g)|}}{k!}\varphi^{\mu_1\cdots\mu_k}\varepsilon_{\mu_1\cdots\mu_n}\\ \Rightarrow (*\varphi)_{\mu_1\ldots\mu_{n-k}} &= \frac{(-1)^s}{k!\sqrt{|\det(g)|}}\varepsilon^{\nu_1\cdots\nu_n}\varphi_{\nu_1\cdots\nu_k}g_{\mu_1\nu_{k+1}}\cdots g_{\mu_{n-k}\nu_n}.\end{aligned}$$

In particular, for an n-form and a 0-form we have

$$\begin{aligned}*\varphi &= \frac{(-1)^s}{\sqrt{|\det(g)|}}\varphi_{1\cdots n},\\ (*\varphi)_{1\ldots n} &= \sqrt{|\det(g)|}\varphi.\end{aligned}$$

△ Recall that some texts (including the first edition of this book) define the Hodge star by the relation $A \wedge C = \langle *A, C \rangle \, \Omega$, in which case these formulas are modified by a factor $(-1)^s$.

Algebra-valued forms

p47, 3.3.5 Algebra-valued exterior forms: These are elements of $\mathfrak{a} \otimes \Lambda^k V^*$. Since the elements of an algebra are vectors, algebra-valued forms may be considered as vector-valued forms whose values can be multiplied. We will reserve the term vector-valued forms for forms whose values are acted on by matrix-valued forms.

Euclidean spaces

p51, 4.1 Generalizing surfaces: $\mathbb{R}^n$ denotes the **Euclidean space** of dimension n, i.e. the manifold of points which are n-tuples of real numbers x^μ with the Euclidean metric; similarly, $\mathbb{R}^{r,s}$ denotes **pseudo-Euclidean space**, the same manifold with the pseudo-Euclidean metric of signature (r, s), while $\mathbb{C}^n$ denotes a complex manifold. This notation applies even if some of the structure is ignored, e.g. if the context is topological spaces, then the metric on $\mathbb{R}^n$ is ignored.

△ A possible source of confusion is the overloading of the above notation for vector spaces, and the status of the origin as a special point, which is left ambiguous. Euclidean spaces are sometimes denoted $\mathbb{E}^n$ to distinguish them from the vector space $\mathbb{R}^n$, but we do not follow this convention since it is less common, the distinction is usually clear from context, and it leaves the status of the origin ambiguous.

Projective lines

p62, 4.3.2 Projective spaces: $\mathbb{R}\mathrm{P}^1$is also called the **real projective line**, and another way of viewing it is to consider the map from each line (omitting the origin) to its slope, with the vertical line then being mapped to infinity, resulting in $\mathbb{R}P^1 \cong \mathbb{R}_\infty \cong S^1$, where $\mathbb{R}_\infty \equiv \mathbb{R} \cup \{\infty\}$ denotes the algebra $\mathbb{R}$ along with a point at infinity:

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R}P^1 \cong \mathbb{R}_\infty \\ \{(x, mx), m, 0 \neq x \in \mathbb{R}\} &\mapsto m \\ \{(0, y), y \in \mathbb{R}\} &\mapsto \infty \end{aligned}$$

Then by generalizing this reasoning we have $\mathbb{C}\mathrm{P}^1 \cong \mathbb{C}_\infty \cong S^2$; $\mathbb{H}\mathrm{P}^1 \cong \mathbb{H}_\infty \cong S^4$; and $\mathbb{O}\mathrm{P}^1 \cong \mathbb{O}_\infty \cong S^8$.

Combining spheres

p63, 4.3.3 Combining spaces: Some facts about combining spheres are:

- If the product $X \times Y = S^n$ then one of the spaces is a point
- The quotient $S^n / S^{n-1} = S^n \vee S^n$ yields a wedge sum
- The suspension $SS^n = S^{n+1}$
- The join $S^n * S^m = S^{m+n+1}$
- The connected sum of n-dimensional manifolds $M \# S^n = M$

n-chains

p68, 5.1.1 Simplices: an n-chain can also be defined as an element of $C_n(X)$, defined to be the free abelian group with basis the n-simplices σ_α. For 0-simplices, we also explicitly define $\partial_0 \sigma = 0$.

Exact sequences

p71, 5.1.4 Chain complexes:

Note that for a chain complex, the image of ∂_{n+1} is contained in the kernel of ∂_n; if these are in fact equal, the chain complex is an **exact sequence**, defined to be any sequence of homomorphisms which sends the image of one object to the kernel of the next. A **short exact sequence** is of the form

$$0 \longrightarrow N \xrightarrow{\phi} E \xrightarrow{\pi} G \longrightarrow 0,$$

and any longer sequence is called a **long exact sequence**. ϕ is injective and π is surjective, so a short exact sequence can be viewed as an embedding of N into E with $G = E/N$. For groups, a short exact sequence is called a **group extension**, or "E is an extension of G by N." Note that N is normal in E since it is the kernel of π, and thus $G \cong E/N$. A **central extension** is one where N also lies in the center of E.

△ A group extension as above is sometimes described as "E is an extension of N by G." A long exact sequence is sometimes defined as any exact sequence that is not short, or as one which is infinite.

Homology examples

p73-74, 5.2.2 Examples:

- To be a bit more explicit, if σ is an n-simplex which encloses the hole and is therefore not a boundary, for every integer a there is a homologically distinct n-chain $a\sigma$ consisting of a copies of σ, with the orientation reversed for negative a.
- In computing $H_2(S^1) = 0$, note that any torus mapped to the circle is the boundary of a solid torus also mapped to the circle.

Points moved by tangent vectors

p85, 6.1.1 Coordinates: Another potential source of confusion is that x^μ is also commonly used to refer to the coordinates of a curve on M.

p86, 6.1.2 Tangent vectors and differential forms: In a given chart, any parametrized curve is defined to have tangent v at $t = 0$ if its coordinates are $C^\mu(t) \equiv a^\mu + tv^\mu$ to first order in t; therefore the coordinates of the tangent vector to C at any point ay be obtained by

$$v^\mu = \frac{\mathrm{d}C^\mu}{\mathrm{d}t}.$$

Anticipating later work, we should also explicitly say here that being able to write $p + \varepsilon v$ allows us to write

$$v(f) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f_{p+\varepsilon v} - f_p \right].$$

Cotangent space and bundle

p87, 6.1.2 Tangent vectors and differential forms: a **covariant vector field** is a 1-form defined in the **cotangent space** $T_p^* U$ at each point such that its value on a vector field is a smooth function, and the set of all cotangent spaces $T^* U$ is called the **cotangent bundle**.

Volume pseudo-form

p88, 6.1.2 Tangent vectors and differential forms: a volume pseudo-form exists on any differentiable manifold, including those which are non-orientable.

Frame synonyms

p89, 6.1.3 Frames: the terms referring to a frame sometimes imply that the frame is orthonormal.

The tangent to a curve

p93, 6.2.2 The differential and pullback: For the parametrized curve $C\colon \mathbb{R} \to N^n$, we define the tangent to the curve at $t \in \mathbb{R}$ to be

$$\begin{aligned} \dot{C}(t) &\equiv \mathrm{d}C\left(\frac{\partial}{\partial x}\right)\bigg|_t \\ &= \frac{\partial C^\lambda}{\partial x}\frac{\partial}{\partial y^\lambda}\bigg|_{C(t)}, \end{aligned}$$

which is also denoted $\frac{\mathrm{d}C(t)}{\mathrm{d}t}$ and coincides with the Euclidean tangent to a curve if $N = \mathbb{R}^n$.

Foliations

A **foliation** of a manifold M^n decomposes it into a union of disjoint connected submanifolds L_j^k called **leaves**, such that there exist coordinates around every point for which the leaves are subspaces with $n-k$ coordinates constant. The **Frobenius theorem** says that a k-dimensional subbundle of TM, a subspace of the tangent space smoothly defined at each point, comprises the tangent bundles of the leaves of a foliation iff the Lie bracket of any two vector fields in the subbundle remains in the subbundle. Two corollaries are that a non-vanishing vector field always defines a flow which is a foliation, and that a frame whose components have vanishing Lie brackets is a coordinate frame for coordinates defined by the orthogonal plane foliations.

△ The quantities defined in the statement of the Frobenius theorem have various names: the tangent subbundle is called a **distribution**, the Lie bracket condition makes the distribution **completely integrable** or **involutive**, and the resulting foliation has leaves which are called **integral manifolds** of the distribution.

△ In general relativity, spacetime is sometimes foliated into space-like leaves defined by a coordinate chart, with the Lorentzian metric split into metrics on each leaf along with quantities relating them; this is referred to as the **3+1 formulation** (AKA 3+1 formalism, 3+1 approach). In this context, the foliation is sometimes called a **slicing**, in which case the leaves are called **slices**.

Submersions

p96, 6.2.4 Critical points: If all points in M are regular then Φ is called a **submersion**.

Lie bracket and Lie derivative

p97-99, 6.3.2 The Lie derivative of a vector field:

- To distinguish from later derivation relations, note that L_v here is so far only a derivation on $\mathrm{vect}(M)$
- The curve $v_p(t)$ is locally unique, and the reference for local flow existence should be Warner p. 37.
- We can also express the Lie derivative as $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[w\left|_p - \mathrm{d}\Phi_\varepsilon\left(w\left|_{v_p(-\varepsilon)}\right.\right)\right.\right]$.
- In considering the second figure, it is helpful to note that $v_p(\varepsilon) = p + \varepsilon v$.

p99-101, 6.3.3 The Lie derivative of an exterior form:

- The explanation of the depiction of 1-forms as "receptacles" is better placed in Section 3.2.1, The structure of the dual space.
- Noting that we can derive a Leibniz rule over contraction $L_v(\varphi(w)) = (L_v\varphi)(w) + \varphi(L_v w)$lets us arrive at an expression for the Lie derivative applied to general tensors, viewed as real-valued mappings on vectors and 1-forms:
$$\begin{aligned} L_v T(\varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n) = v\left(T(\varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)\right) \\ - \sum_{j=1}^{m} T\left(\varphi_1, \ldots, L_v\varphi_j, \ldots, \varphi_m, w_1, \ldots, w_n\right) \\ - \sum_{j=1}^{n} T\left(\varphi_1, \ldots, \varphi_m, w_1, \ldots, L_v w_j, \ldots, w_n\right) \end{aligned}$$
In a holonomic frame, this yields the expression for the Lie derivative of a tensor in terms of coordinates, which for consistency in indices can be written
$$\begin{aligned} L_v T^{\mu_1\ldots\mu_m}{}_{\sigma_1\ldots\sigma_n} &= v^\lambda \frac{\partial}{\partial x^\lambda} T^{\mu_1\ldots\mu_m}{}_{\sigma_1\ldots\sigma_n} \\ &- \sum_{j=1}^{m} \left(\frac{\partial v^{\mu_j}}{\partial x^\lambda}\right) T^{\mu_1\ldots\mu_{j-1}\lambda\mu_{j+1}\ldots\mu_m}{}_{\sigma_1\ldots\sigma_n} \\ &+ \sum_{j=1}^{n} \left(\frac{\partial v^\lambda}{\partial x^{\sigma_j}}\right) T^{\mu_1\ldots\mu_m}{}_{\sigma_1\ldots\sigma_{j-1}\lambda\sigma_{j+1}\ldots\sigma_n}. \end{aligned}$$
From this we can confirm that the Lie derivative satisfies the Leibniz rule over the tensor product, and therefore is a derivation of degree 0 on both the tensor algebra and the exterior algebra.

Exterior derivative coordinate expressions

p105, 6.3.5: The exterior derivative of a k-form: In a holonomic frame, we can obtain an expression for $\mathrm{d}\varphi$ in terms of coordinates

$$\begin{aligned} \mathrm{d}\varphi &= \sum_{\mu_0<\cdots<\mu_k} \left(\sum_{j=0}^{k} (-1)^j \frac{\partial}{\partial x^{\mu_j}} \varphi_{\mu_0\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_k} \right) \mathrm{d}x^{\mu_0} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \\ &= \frac{\partial}{\partial x^{\mu_0}} \varphi_{\mu_1\ldots\mu_k} \mathrm{d}x^{\mu_0} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \\ &= \frac{\partial \varphi_I}{\partial x^{\mu_0}} \mathrm{d}x^{\mu_0} \wedge \mathrm{d}x^I, \end{aligned}$$

so that in terms of array components we have

$$(\mathrm{d}\varphi)_{\mu_0\ldots\mu_k} = \sum_{j=0}^{k} (-1)^j \frac{\partial}{\partial x^{\mu_j}} \varphi_{\mu_0\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_k}.$$

Generalized divergence

p105, 6.3.5: The exterior derivative of a k-form:

The generalizations of vector calculus can be extended to a pseudo inner product with signature (r, s) by defining the divergence as $(-1)^s * \mathrm{d}(*\varphi)$, which is then independent of both signature and orientation.

p106, 6.3.5 The exterior derivative of a k-form: We can further generalize the divergence to k-forms φ by defining the **codifferential** (AKA coderivative, exterior coderivative)

$$\begin{aligned} \delta\varphi &\equiv (-1)^k *^{-1} \mathrm{d}(*\varphi) \\ &= (-1)^{n(k+1)+s+1} * \mathrm{d}(*\varphi). \end{aligned}$$

The map $\delta : \Lambda^k M \to \Lambda^{k-1} M$ does not follow the Leibniz rule and so is not a derivation. However, we do have $\delta^2 = 0$, so that we may write $\Delta \equiv (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}$, which (usually for $s = 0$) is called the **Laplace-Beltrami operator** (AKA Laplace operator, Laplacian, Laplace–de Rham operator); a form on a Riemannian manifold for which it vanishes is called a **harmonic form**. For $f \in \Lambda^0 M$ we have $\delta f = 0$ and $\Delta f = -\mathrm{div}\,(\nabla f)$; on a flat Lorentzian manifold in the mostly pluses signature its negative is denoted $\Box f \equiv \partial^\mu \partial_\mu f = -\partial_t^2 f + \Delta f$, where $\Box$ is called the **d'Alembertian** (AKA d'Alembert operator, wave operator, box operator), whose spatial part despite the opposite sign is denoted $\Delta f \equiv \nabla^2 f \equiv \nabla \cdot \nabla f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$ and called the **Laplacian** (AKA Laplace operator). $\Delta f = 0$ is then called **Laplace's equation**, while a fixed $\rho \in \Lambda^0 M$ defines **Poisson's equation** $\Delta f = \rho$.

> △ The operators above may be defined with a negative sign, in particular in the mostly minuses signature on a Lorentzian manifold, and depending upon whether applied to a function or a form.

If $\varphi \in \Lambda^k M$ and $\psi \in \Lambda^{k+1} M$ so that $\mathrm{d}\,(\varphi \wedge *\psi) \in \Lambda^n M$, it is not hard to see that

$$\int_{\partial M} (\varphi \wedge *\psi) = \int_M \langle \mathrm{d}\varphi, \psi \rangle \, \Omega - \langle \varphi, \delta\psi \rangle \, \Omega,$$

which means that if $\varphi \wedge *\psi$ vanishes on ∂M (or $\partial M = 0$) we have $\int_M \langle \mathrm{d}\varphi, \psi \rangle \, \Omega = \int_M \langle \varphi, \delta\psi \rangle \, \Omega$. In particular, for $f \in \Lambda^0 M$ and $v^\flat \in \Lambda^1 M$, we have

$$\int_{\partial M} f \left(*v^\flat \right) = \int_M \langle \nabla f, v \rangle \, \Omega + f \mathrm{div}\,(v)\, \Omega,$$

or for $f = 1$ and recalling from Section **??** that $i_v \Omega = (-1)^s * (v^\flat)$,

$$\begin{aligned} \int_{\partial M} *v^\flat &= \int_M \mathrm{div}\,(v)\, \Omega \\ \Rightarrow \int_M \mathrm{div}\,(v)\, \Omega &= \int_{\partial M} i_v \Omega \\ &= \int_{\partial M} \langle v, \hat{n} \rangle \, i_{\hat{n}} \Omega, \end{aligned}$$

where $\hat{n}$ is the unit normal vector to ∂M, the classical **divergence theorem**.

Cartan's formula

p107, 6.3.6 Relationships between derivations: The last relation is sometimes called **Cartan's formula** (AKA Cartan's magic formula).

Lie algebra of a Lie groups

p114, 7.2.1 The Lie algebra of a Lie group: The defining equation of left invariance is more clearly written $\mathrm{d}L_g\left(A|_h \right) = A|_{L_g(h)} = A|_{gh}$. Also, it's helpful to remember that the isomorphism from $\mathfrak{g}$ to $T_e G$ is as a vector space, with vector multiplication dependent upon the vector fields determined by the elements of this vector space.

Baker-Campbell-Hausdorff formula

p116, 7.2.2 The Lie groups of a Lie algebra: As stated, the Baker-Campbell-Hausdorff formula is valid for the exponential map for all associative algebras whose Lie commutator give the Lie bracket, but it is more general to note that in terms of Lie brackets it holds for the exponential map of any Lie algebra. It is also important to note that the series may not converge, limiting validity to a neighborhood of the identity.

Linear algebra

p121, 7.3.2 Linear algebra:

- The adjoint is also known as the conjugate transpose
- In terms of the associated abstract linear transformation, the matrix **rank** is the dimension of its image and the adjoint is defined by $\langle v, Aw\rangle = \left\langle A^{\dagger}v, w\right\rangle$.
- Recalling the section on combinatorial notations, $\det(A)$ is the volume change multiple associated with A applied to an orthonormal basis, while $\mathrm{tr}(A)$ is then the volume change addition per unit t of $\exp(tA)$.
- The tensor product of hermitian / unitary matrices is hermitian / unitary
- Another basic but non-obvious fact: diagonalizable matrices commute iff they are simultaneously diagonalizable
- The spectral theorem is more completely described as saying that a matrix is normal iff it can be diagonalized by a unitary similarity transformation; a real matrix is symmetric iff it can be diagonalized by an orthogonal similarity transformation
- A similarity transformation causes the matrix representation of the form in the old basis to become $\left(B^{-1}v\right)^{\mathrm{T}}\left(B^{\mathrm{T}}\varphi B\right)\left(B^{-1}w\right)$ $v^{\mathrm{T}}\varphi w$, where the matrices $B^{\mathrm{T}}\varphi B$ and φ are called **congruent**.

Matrix group terminology

p126, 7.3.6 Matrix group terminology in physics: The (homogeneous) Lorentz group is also called the homogeneous Poincaré group.

Torsors

p129, 7.4.1 Group actions: These concepts highlight potential confusions around $\mathbb{R}^n$ as a vector space, Lie group, and manifold: the action of the Lie group of vectors under addition $\mathbb{R}^n$ on the manifold $\mathbb{R}^n$ by displacement is regular, making the manifold a $\mathbb{R}^n$-torsor, i.e. it is isomorphic to the Lie group $\mathbb{R}^n$ but with the origin forgotten.

Classification of Lie groups and algebras

p139, 7.5.2 Simple Lie algebras: The complex simple Lie algebra subscripts correspond to the **Lie algebra rank**, whose definition we omit.

p140, 7.5.3 Classifying representations: In one form of classification used in physics, the finite-dimensional complex irreps of a finite-dimensional Lie algebra $\mathfrak{g}$ may be characterized or labeled using the concept of a **Casimir element**. A Casimir element E is an element of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, which as we recall is the associative algebra which includes $\mathfrak{g}$ as a subalgebra under the Lie commutator. A Casimir element E is constructed from a basis of $\mathfrak{g}$ and a nondegenerate bilinear form B (which E is dependent upon); B is also required to be "invariant" (AKA Ad-invariant, ad-invariant, associative), meaning that

$$B\left([u, w], v\right) = B\left(u, [w, v]\right),$$

which if $\mathfrak{g}$ is the Lie algebra of a connected Lie group G implies invariance under the adjoint rep, i.e. $\forall g \in G$

$$B\left(g_{\mathrm{Ad}}(u), g_{\mathrm{Ad}}(v)\right) = B(u, v).$$

The key property of a Casimir element is that it can be shown to be an element of the center of $U(\mathfrak{g})$ (commutes with all elements), which by Schur's Lemma means that under any finite-dimensional complex irrep ρ the **Casimir operator** $\rho(E)$ is a scalar multiple of the identity matrix $C(\rho)I$, where the scalar $C(\rho)$ (which is often described as the eigenvalue of the Casimir operator) is called the **Casimir invariant**, and may then be used to label the irrep.

△ $\rho(E)$ denotes E constructed from the basis vector irreps under ρ. Note that the Casimir invariant for a given B is not necessarily different for each irrep.

△ The terms Casimir element, Casimir operator, and Casimir invariant are often used interchangeably.

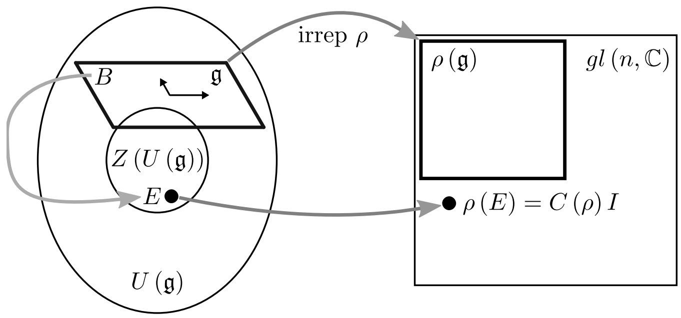

FIGURE 3: The Casimir element is constructed from an invariant nondegenerate bilinear form B on $\mathfrak{g}$, and is an element of the center of the universal enveloping algebra $U(\mathfrak{g})$. It maps to the Casimir operator $\rho(E)$ under any finite-dimensional complex irrep ρ of $\mathfrak{g}$, which by Schur's Lemma is the identity matrix I multiplied by the Casimir invariant $C(\rho)$.

Now, there exists an symmetric invariant bilinear form called the **Killing form** (AKA Cartan-Killing form) which may be defined on any Lie algebra; it is nondegenerate iff the Lie algebra is semisimple (a fact called the **Cartan criterion**), which makes it an invariant pseudo inner product $\langle\rangle_K$ on a real semisimple Lie algebra. For a real semisimple Lie algebra, we may therefore choose a basis $\hat{e}_j$ which is orthonormal under the Killing form, and which can be used to construct a specific Casimir element called the quadratic (AKA second order) Casimir element

$$e^2 \equiv \sum_j \hat{e}_j \hat{e}_j.$$

△ A Lie algebra including an invariant pseudo inner product is sometimes called a metric (AKA orthogonal, quadratic, self-dual) Lie algebra.

Furthermore, for a simple Lie algebra, it can be shown that any symmetric invariant bilinear form is proportional to the Killing form. Since the trace is a bilinear form on multiplied matrices, which can be verified to be symmetric and invariant due to its cyclic property, we may therefore define the **Dynkin index** (AKA index, second order Dynkin index) $Y(\rho)$ of a finite-dimensional complex rep ρ by

$$\begin{aligned} \operatorname{tr}(\rho(v)\,\rho(w)) &= Y(\rho)\,\langle v,w\rangle_K \\ \Rightarrow \operatorname{tr}(\rho(\hat{e}_j)\,\rho(\hat{e}_k)) &= Y(\rho)\,\delta_{jk}. \end{aligned}$$

Taking the trace of the quadratic Casimir operator under a finite-dimensional complex irrep then yields a relationship in terms of the vector space dimensions of $\mathfrak{g}$ and the space acted on by the rep ρ:

$$\begin{aligned} \operatorname{tr}\left(\sum_j \rho(\hat{e}_j)\,\rho(\hat{e}_j)\right) &= Y(\rho)\dim(\mathfrak{g}) \\ &= \operatorname{tr}(C(\rho)\,I) = C(\rho)\dim(\rho) \\ \Rightarrow C(\rho) &= \frac{Y(\rho)\dim(\mathfrak{g})}{\dim(\rho)} \end{aligned}$$

△ Note that although for a simple Lie algebra any symmetric invariant bilinear form is proportional to the Killing form, there may be other nondegenerate invariant bilinear forms which can be used to construct other Casimir elements. In fact, it can be shown that the number of algebraically independent Casimir elements for a simple Lie algebra is equal to its rank; these elements algebraically generate the center of the algebra. It can also be shown that the negative of the Killing form is a (positive definite) inner product iff the real semisimple Lie algebra is a compact real form, and that the Killing form is actually invariant under any automorphism of $\mathfrak{g}$.

The main physical application of all this is to label simple algebra reps by the eigenvalues of the rep of e^2. In particular, in quantum physics, the state of a physical system is associated with a vector in a complex Hilbert space. The laws of physics are equations which are postulated to be invariant under various symmetry transformations, in particular those of $SU(n)$ and $SO(r,s)$, and hence their (simple) Lie algebras. The Hilbert space must therefore be acted on by ("carry") a rep of each of these Lie algebras, under which the state vector is transformed in a way which leaves the laws invariant. Recalling Weyl's theorem, these simple Lie algebras have completely reducible reps, and the subspace of state vectors acted on by a component irrep is assumed to correspond to a single elementary particle, which may then be labeled by its quadratic Casimir invariant.

☼ Note that for a connected G whose Lie algebra is $\mathfrak{g}$, $\langle\rangle_K$ is invariant under the adjoint rep, which for a matrix group is a similarity transformation, i.e.

$$\begin{aligned}\left\langle g_{\mathrm{Ad}}(u), g_{\mathrm{Ad}}(v)\right\rangle_K &= \langle u, v\rangle_K \\ &= \left\langle gug^{-1}, gvg^{-1}\right\rangle_K .\end{aligned}$$

Thus for the groups $SU(n)$ or $SO^e(r,s)$, which in gauge theories can be viewed as passive "rotations" (coordinate transformations preserving an inner product) on the vector space being acted upon, the Killing form (and therefore the trace) is a pseudo inner product on any $\mathfrak{g}$-valued form which is independent of these coordinates, as one would naturally want to require. In fact, for these simple Lie algebras it is the unique invariant inner product up to a choice of units (since all others are proportional).

Complexified Clifford algebras

p143, 8.1.1 Isomorphisms: Note that when complexifying, we have e.g. $(iv)(iw) = \langle iv, iw\rangle = -vw = -\langle v, w\rangle$, so that $\langle\rangle$ is not a complex inner product as we have defined it; in particular, $\langle v, v\rangle$ can have an imaginary part.

Clifford algebra representations

p143-145, 8.1.2 Representations and spinors: The classification of faithful Clifford algebra irreps using matrix algebra isomorphisms also reveals whether these various irreps are faithful. In even dimension, neither $C(r,s)$ nor $C^{\mathbb{C}}(n)$ is a direct sum, so the Dirac irrep is faithful. In odd dimension, for $r-s=1$ or 5 mod 8, $C^{\mathbb{C}}(n)$ is a direct sum but so is $C(r,s)$, so neither Dirac irrep is faithful; when $r-s=3$ or 7 mod 8, $C^{\mathbb{C}}(n)$ is a direct sum but $C(r,s)$ is not, and therefore the two Dirac irreps are faithful. The Majorana and Majorana-Weyl reps are isomorphisms, and hence are faithful iff they are not a direct sum. When $r-s=0$ mod 8, $C_0(r,s)$ is a direct sum $\mathbb{R}(m)\oplus\mathbb{R}(m)$ which sits in $C^{\mathbb{C}}(n-1)\cong\mathbb{C}(m)\oplus\mathbb{C}(m)$, where $m\equiv 2^{(n-2)/2}$, so the Weyl irreps are unfaithful, as they are for $r-s=4$ mod 8. When $r-s=2$ mod 8 (including for $C_0(3,1)$), $C_0(r,s)\cong\mathbb{C}(m)$ sits in $\mathbb{C}(m)\oplus\mathbb{C}(m)$, so the Weyl irreps are faithful, as they are for $r-s=6$ mod 8.

Dirac matrices

p145-148, 8.1.3 Pauli and Dirac matrices: A chiral basis is defined by a block diagonal decomposition of $C_0(3,1)$ into two chiral Weyl reps. We know that our Majorana matrices act as a basis due to **Pauli's fundamental theorem**, whose extended form states that for even $r+s=n$, any two sets of n anti-commuting elements of $C(r,s)$ which square to ±1 according to the signature are related by a similarity transformation; this means that any such elements can act as a basis for the vector space generating the Clifford algebra, since one of them must. This theorem also holds for $C^{\mathbb{C}}(n)$ for even n. Also, the complexified algebra $C^{\mathbb{C}}(4) \cong \mathbb{C}(4)$ is sometimes called the **Dirac algebra**, and in $C(4,1) \cong \mathbb{C}(4)$ we may choose a basis consisting of Dirac matrices from $C(3,1)$ and $\gamma^4 \in C(4,1) \equiv \gamma_5 \in C(3,1)$. Finally, both $SU(2)$ and $SO(3)$ can be written $\exp\left(ia^j\sigma_j\right)$ since they are both compact connected Lie groups.

Chiral decomposition

p149, 8.1.4 Chiral decomposition: The decomposition by orthogonal projection is maintained under Clifford multiplication since the unit n-vector Ω of $C\,(r,s)$ commutes with any $A \in C_0\,(r,s)$. We can see that the chiral basis Dirac matrices from the previous section block diagonalize $C_0(3,1)$ due to the form of γ_5; this also allows us write a Dirac spinor $\psi \in \mathbb{C}^4$ as stacked Weyl spinors, which explains why these are called half-spinors. The stacked decomposition into half-spinors remains invariant under the transformation $\psi \rightarrow A\psi$ by any $A \in C_0\,(r,s)$. Under parity, the chiral spinor reps are swapped: $P_\pm \rightarrow P_\mp \Rightarrow \psi_\text{L} \leftrightarrow \psi_\text{R}$. The parity operation reverses spatial orientation as a mirror image does, which is why the half-spinors are called chiral: they are swapped under parity, just as the right and left hands are swapped in a mirror image.

Clifford groups and rotations

p149, 8.2.1 Reflections:

In the figure, define $\hat{v}_\perp$ to be the unit vector perpendicular to u; then

$$u' \equiv u\cos\frac{\theta}{2} + \hat{v}_\perp \sin\frac{\theta}{2}$$

is a vector rotated by $\theta/2$ from u. The combination of reflections using these two vectors yields a rotation of v by θ in the $u \wedge v$ plane:

$$\begin{aligned} R_\theta\,(v) &= u'uvuu' \\ &= (u'u)\,v\,(u'u)^{-1} \end{aligned}$$

Note that for infinitesimal θ, we then have

$$\begin{aligned} u'u &= 1 + \hat{v}_\perp u\theta/2 \\ &= \exp\left(\hat{v}_\perp u\theta/2\right), \end{aligned}$$

so that an infinitesimal rotation corresponds to the exponential of a bivector.

p150, 8.2.2 Rotations:

- Distinctions are sometimes made between (special) Clifford groups and (S)Pin groups.
- Any element of the orthogonal group $O\,(r,s)$ is a rotation and/or reflection, and the **Cartan–Dieudonné theorem** states that any such transformation can be obtained as a product of at most $r+s$ reflections.

Spin group properties

p152, 8.2.3 Lie group properties: $\text{Pin}(r,s)$ is sometimes defined as the Clifford product of "unit" vectors u with only positive $\langle u,u\rangle = +1$. Then $\text{Spin}(r,s)$ ends up being the double cover of $SO(r,s)^e$, i.e. it is what we call $\text{Spin}(r,s)^e$ for most cases: it is connected for r or s greater than one, and is simply connected if $s = 0,1$ and $r > 2$ (or vice versa).

Specific spin group elements

p155, 8.2.4 Lorentz transformations: In 3 dimensional space, if we start with the transformation instead of the basis, we can express it as the exponential of a single 2-blade in the plane of rotation.

Spacetime spin reps

p156-7, 8.2.5 Representations in spacetime: The presentation is better expressed as results of the Dirac matrices, rather than guesses which happen to match them.

Vectors in a plane as complex numbers

p159, 8.2.6 Spacetime and spinors in geometric algebra: The vectors in any space-like plane in a vector space can be identified with the complex numbers via the representation of the isomorphism $C_0(2,0) \cong C(0,1) \cong \mathbb{C}$ effected by $\hat{e}_1\hat{e}_2 \equiv \Omega \to i$, where i is the unit vector in $C(0,1)$ identified with the imaginary unit in $\mathbb{C}$. A vector $v \equiv v^1\hat{e}_1 + v^2\hat{e}_2 \in C(2,0)$ is represented by $v_{\mathrm{E}} \equiv \hat{e}_1 v = v^1 + v^2\hat{e}_1\hat{e}_2$ in the even subalgebra $C_0(2,0)$, and therefore by $v_{\mathbb{C}} = v^1 + iv^2$ in $\mathbb{C}$, where the choice of $\hat{e}_1$ thus defines the real axis. Complex conjugation is then the reflection across the imaginary axis $\hat{e}_1 v_{\mathrm{E}}\hat{e}_1 = v\hat{e}_1 = \tilde{v}_{\mathrm{E}} = v^1 + v^2\hat{e}_2\hat{e}_1 = v^1 - iv^2 = v^*_{\mathbb{C}}$, which is also reversion in $C_0(2,0)$. The complex inner product is $\langle v_{\mathbb{C}}, w_{\mathbb{C}}\rangle_{\mathbb{C}} = v^*_{\mathbb{C}} w_{\mathbb{C}} = \tilde{v}_{\mathrm{E}} w_{\mathrm{E}} = \langle v, w\rangle_{\mathbb{R}}$. Note that multiplication by the imaginary unit in $\mathbb{C}$ is represented by right Clifford multiplication by Ω in $C_0(2,0)$: $v_{\mathrm{E}}\Omega = \hat{e}_1 v\Omega = v^1\Omega - v^2 = iv^1 - v^2 = iv_{\mathbb{C}}$. This means that exponential rotations must also act from the right in $C_0(2,0)$, and since Ω anti-commutes with vectors, both operations from the left reverse sign: $e^{\Omega\theta} v_{\mathrm{E}} = e^{\Omega\theta}(\hat{e}_1 v) = \hat{e}_1\left(e^{-\Omega\theta} v\right) = \left(e^{-\Omega\theta} v\right)_{\mathrm{E}} = e^{-i\theta} v_{\mathbb{C}}$.

The covariant derivative in terms of the connection

p169, 9.1.5 The covariant derivative in terms of the connection: The notation for partial derivatives is extended to $\partial_v f \equiv v^a \partial_a f$. Another potentially confusing aspect is that $\partial^a \equiv g^{ab}\partial_b$ is not the frame dual to ∂_a, which is $\mathrm{d}x^a$. Another is that $\partial_a f$ is a 1-form, but ∂_a alone is not. Another is that when using a coordinate frame based on curvilinear coordinates in Euclidean space, parallel transport is implicit in taking partial derivatives of vectors, resulting in the expression $\partial_\mu e_\lambda = e_\sigma \Gamma^\sigma{}_{\lambda\mu}$.

Geodesics

p170, 9.1.7 Geodesics and normal coordinates: To distinguish it from the distance minimizing geodesic of a Riemannian manifold, a geodesic defined by parallel transported tangent vectors is called an **autoparallel geodesic**. Expressing a geodesic as a parametrized curve $C^\mu(t)$ with tangent $v^\mu(t) \equiv \dot{C}^\mu(t)$ in given coordinates, we can write

$$\begin{aligned}
\nabla_v v &= v^\lambda\left(\partial_\lambda v^\mu + \Gamma^\mu{}_{\sigma\lambda} v^\sigma\right) \\
&= \partial_v\left(v^\mu\right) + \Gamma^\mu{}_{\sigma\lambda} v^\sigma v^\lambda \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}C^\mu}{\mathrm{d}t}\right) + \Gamma^\mu{}_{\sigma\lambda}\frac{\mathrm{d}C^\sigma}{\mathrm{d}t}\frac{\mathrm{d}C^\lambda}{\mathrm{d}t} \\
&= \frac{\mathrm{d}^2 C^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\sigma\lambda}\frac{\mathrm{d}C^\sigma}{\mathrm{d}t}\frac{\mathrm{d}C^\lambda}{\mathrm{d}t} = 0,
\end{aligned}$$

where the last line is called the **geodesic equation**, and in the third line we use the fact that the change of the 0-form v^μ in the v direction is equal to the derivative of the function $v^\mu(t)$ with respect to t.

The covariant derivative on the tensor algebra

p175, 9.2.1 The covariant derivative on the tensor algebra: it is worth noting that for functions, $\nabla_v f = L_v f = \mathrm{d}f$. It is also important to remember that we may raise and lower indices in a covariant derivative expression, since $\nabla_c g_{ab} = 0 \Rightarrow g^{ad}\nabla_c T_{ab} = \nabla_c T^d{}_b$, but we may not do so in a partial derivative expression: $g^{ad}\partial_c T_{ab} \neq \partial_c T^d{}_b$.

Torsion

p183, 9.2.4 Torsion: More explicitly, the zero torsion expression $[v,w] = \nabla_v w - \nabla_w v$ means that we can replace partial with covariant derivatives in the usual expression for the Lie derivative of a vector field:

$$\begin{aligned}(L_v w)^a &= [v,w]^a \\ &= v^b \partial_b w^a - w^b \partial_b v^a \\ &\overset{\mathcal{T}}{=} v^b \nabla_b w^a - w^b \nabla_b v^a\end{aligned}$$

Second Bianchi identity

p190-2, 9.2.7 Second Bianchi identity: This is also known as the differential Bianchi identity.

The Riemannian metric

p194, 9.3.1 The Riemannian metric:

- The metric and manifold are also described by the same terms used to characterize the signature, i.e. Lorentzian manifold, Minkowskian metric, etc.
- The minimum length curve connecting two points is called a **Riemannian geodesic**. It can be shown that for any tangent vector v on a Riemannian manifold there is a unique geodesic $C_v(d)$ parametrized by distance whose tangent is v; one can then define the exponential map by $\exp(v) \equiv C_v(1)$.
- The volume pseudo-form can be integrated over non-orientable regions, and it along with the volume element and form are defined here relative to a specified metric.

△ A potential source of confusion is that the terms isometry and isometric are also used to refer to maps between metric spaces which preserve the distance function. For example, S^2 can be isometrically embedded in $\mathbb{R}^3$ with regard to the Riemannian metric, but not with regard to the distance function.

The Levi-Civita connection

p197-8, 9.3.2 The Levi-Civita connection:

- It is clearer to directly note that the parallel transport of tensors just transports the arguments, so we have $(\|_{-C}\, g_{ab})\, v^a w^b = g_{ab} \|_C\, v^a \|_C\, w^b$, and hence $\|_{-C}\, g_{ab} = g_{ab}$, or $\nabla_c g_{ab} = 0$.
- $\nabla_c g_{ab} = 0$ implies that $\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}$, and considering $\partial_c \left(g^{ad} g_{df}\right) = \partial_c \left(\delta^a{}_f\right) = 0$, we arrive at the complementary expression $\partial_c g^{ab} = -g^{ad} g^{bf} \partial_c g_{df} = -\left(\Gamma^{ab}{}_c + \Gamma^{ba}{}_c\right)$.
- The geodesics defined by the parallel transport associated with the Levi-Civita connection can be shown to be exactly those defined by the metric.
- The connection determining the metric only requires M to be connected (not necessarily simply connected).
- Given a connected manifold M with a torsion-free connection, a metric of signature (r,s) compatible with this connection exists if and only if $\mathrm{Hol}(M) \subseteq O(r,s)$; in fact, any metric which is invariant under $\mathrm{Hol}(M)$ is such a compatible metric.
- We will denote the Levi-Civita connection and related quantities with an overbar, e.g. $\overline{\Gamma}$, $\overline{\nabla}$, and $\overline{R}$.
- In the case of a general connection where $\mathrm{Hol}(M) = SO(n)$ on an orientable M, the associated metric is unique up to a scaling factor (in physics, this corresponds to a choice of units). If $\mathrm{Hol}(M) = SO(n-1,1)$, then there is a separate scaling factor for the time-like component. If $\mathrm{Hol}(M)$ is the identity, then the connection is flat, and the metric is determined by a choice of inner product (of any signature) at any point, which applies to all other points via the unique parallel transport.

- The relation $\overline{R}_{cdab} = \overline{R}_{abcd}$ uses the first Bianchi identity for zero torsion.
- The derivation of the Koszul formula also uses the zero torsion relation $\nabla_v w = \nabla_w v + [v, w]$.
- The Christoffel symbols are sometimes denoted $\left\{ {\lambda \atop \mu\sigma} \right\}$ or $\left\{ {\mu\sigma \atop \lambda} \right\}$.

Independent quantities and dependencies

p198, 9.3.3 Independent quantities and dependencies: From their definitions, the parallel transport and connection in general determine each other. It can be shown that every manifold admits a connection, and every other connection can be obtained by adding a frame-independent $gl\,(\mathbb{R}^n)$-valued 1-form (tensor field of type $(1, 2)$) to it. A connection with $\mathrm{Hol}(M) \subseteq O(r, s)$ may not be able to be defined on a given manifold, but if it can, then given this connection (apart from special cases) it uniquely determines a compatible signature (r, s) metric (up to constant scaling factors). The connection can then be written as the sum of the Levi-Civita connection $\overline{\Gamma}$ of this metric and a tensor K called the **contorsion tensor** (AKA contortion tensor), which using previous relations can be expressed in terms of the torsion:

$$\begin{aligned} \Gamma &= \overline{\Gamma} + K \\ K_{abc} &= \frac{1}{2}\left(T_{bac} + T_{cab} - T_{abc}\right) \\ \Rightarrow T_{abc} &= K_{acb} - K_{abc} \end{aligned}$$

It is important to note that the definition of contorsion incorporates the metric via the lowered index; thus for example changing $\overline{\Gamma}$ (and thus the metric) while holding K constant alters T. Also note that K is anti-symmetric in its first two indices since T is anti-symmetric in its last two. The geodesic equation is

$$\begin{aligned} 0 = (\nabla_v v)^a &= \left(\overline{\nabla}_v v\right)^a + K^a{}_{bc} v^b v^c \\ &= \left(\overline{\nabla}_v v\right)^a - T_{bc}{}^a v^b v^c, \end{aligned}$$

so that the autoparallel geodesics of a connection with non-zero torsion coincide with the Riemannian geodesics of the Levi-Civita connection iff K (and therefore T) is completely anti-symmetric. This can only occur in at least three dimensions, where in an orthonormal frame the anti-symmetry indicates that parallel transport "spins" as in the last figure in the section on torsion.

△ If one uses the convention which reverses the lower indices of the connection coefficients, the same happens to the contorsion tensor to yield $K_{abc} = \frac{1}{2}\left(T_{bac} + T_{cab} + T_{abc}\right)$. In the literature one finds authors who use our convention for the lower indices of the connection coefficients, but reverse the sign of the torsion tensor and/or halve it, which sign reverses and/or doubles our expression for the contorsion tensor.

△ Given a connection $\Gamma^\lambda{}_{\mu\sigma}$ in a coordinate frame with associated metric g, it is important to remember that while the anti-symmetric part of Γ is proportional to the torsion, the symmetric part is not the Levi-Civita connection for g (although it is for some other metric, and has the same autoparallel geodesics as Γ). In particular, one sometimes sees statements to the effect that since only the symmetric part of Γ affects the geodesic equation, "torsion does not affect geodesics." What is true is that subtracting the torsion from Γ leaves autoparallel geodesics unchanged (but does change the metric and hence Riemannian geodesics). But it is important to remember that the more geometric notion of modifying the torsion of Γ (leaving $\overline{\Gamma}$ and thus g unchanged) adds a symmetric part to Γ which does in fact change geodesics.

If the curvature is given over M, there is at most one metric (also apart from special cases, up to a scaling factor, and for $n > 2$) whose Levi-Civita connection yields this curvature.

The divergence and useful relations

p200, 9.3.4 The divergence and conserved quantities: Recall that the divergence of a vector field u can be generalized to a pseudo-Riemannian manifold of signature (r, s) by defining $\mathrm{div}(u) \equiv (-1)^s * \mathrm{d}(*(u^\flat))$. Also recalling that $i_u\Omega = *(u^\flat)$ and $(-1)^s A = (*A)\Omega$ for $A \in \Lambda^n M^n$, we have $\mathrm{d}(i_u\Omega) = \mathrm{d}(*(u^\flat)) = (-1)^s * \mathrm{d}(*(u^\flat))\Omega = \mathrm{div}(u)\Omega$.

Note that both the coordinate- and metric-dependent expression for the divergence and the expression $\nabla_a u^a$ (sometimes called the **covariant divergence**) in terms of the Levi-Civita connection are coordinate-independent and equal to $\partial_a u^a$ in Riemann normal coordinates, confirming our expectation that for zero torsion we have

$$\mathrm{div}(u) = \overline{\nabla}_a u^a.$$

Recall however that the connection coefficients do not vanish in Riemann normal coordinates for non-zero torsion; in this case we can use the contorsion tensor contraction $K^a{}_{ba} = T^a{}_{ab}$ to relate the pseudo-Riemannian divergence to the covariant divergence by

$$\mathrm{div}(u) = \nabla_a u^a - T^a{}_{ab} u^b.$$

△ Note that the different symbols and names given here for the pseudo-Riemannian divergence versus the covariant divergence are oftentimes not distinguished, since they are the same for zero torsion. The distinction also vanishes if the torsion is completely anti-symmetric, i.e. if it leaves geodesics unchanged.

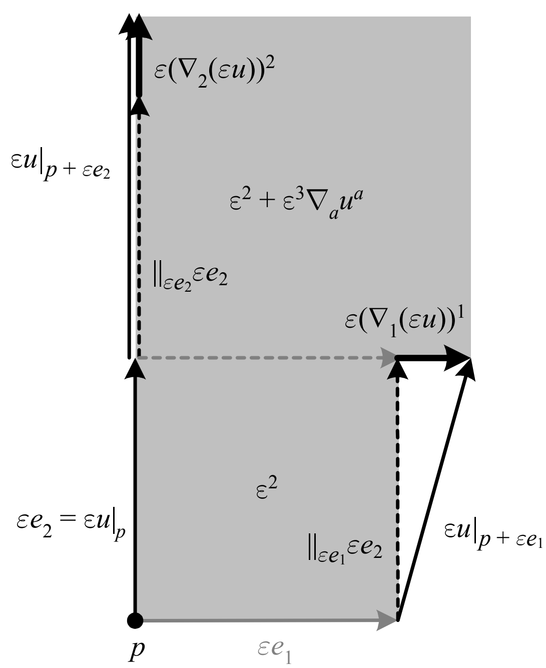

FIGURE 4: The divergence measures the change in volume due to the flow. Here we assume zero torsion, and that the vector field u has unit length at point p, and choose an orthonormal frame which aligns e_2 with u. Each covariant derivative extends a face of the volume, with their sum being proportional to the total change in volume. Note that the upper right corner is of order ε^4 and so can be neglected, and e.g. any component of $\nabla_1 u$ orthogonal to e_1 leaves the volume unchanged, since a more accurate depiction would include the volume with edge $-\varepsilon e_1$, where by linearity this component would be in the opposite direction and thus cancel the volume change. Also note that non-zero torsion would reduce the top edge $\|_{\varepsilon e_2} \varepsilon e_1$ by $\varepsilon^2 T^1{}_{1b} u^b$, which must be added back by subtracting this component, matching the algebraic result.

Note that even in the presence of curvature, the continuity equation holds for the components of the coordinate-dependent quantity $\mathfrak{J} \equiv J\sqrt{|\det(g)|}$, since

$$\begin{aligned} \partial_\mu \mathfrak{J}^\mu &= \partial_t \mathfrak{J}^t + \partial_i \mathfrak{J}^i \\ &= \partial_t \mathfrak{J}^t + \overline{\nabla}_i \mathfrak{J}^i = 0. \end{aligned}$$

Coordinate and tensor divergences

p200, 9.3.4 The divergence and conserved quantities: Using previous results and the relation $\overline{\Gamma}^\mu{}_{\lambda\mu} = \Gamma^\mu{}_{\lambda\mu} - T^\mu{}_{\mu\lambda}$, we can derive many useful coordinate dependent relations. Adopting the common abbreviation

$$\sqrt{g} \equiv \sqrt{|\det(g_{\mu\nu})|}$$

and including torsion for completeness, we expand both sides of the coordinate divergence expression to get

$$\begin{aligned}\partial_\lambda\sqrt{g} &= \sqrt{g}\left(\Gamma^\mu{}_{\lambda\mu} - T^\mu{}_{\mu\lambda}\right)\\ &= \sqrt{g}\,\overline{\Gamma}^\mu{}_{\lambda\mu},\end{aligned}$$

which along with the expressions for the metric derivative from the section on the Levi-Civita connection yields

$$\begin{aligned}\partial_\lambda\left(\sqrt{g}g^{\mu\nu}\right) &= \sqrt{g}\left(g^{\mu\nu}\Gamma^\sigma{}_{\lambda\sigma} - g^{\mu\nu}T^\sigma{}_{\sigma\lambda} - \Gamma^{\mu\nu}{}_\lambda - \Gamma^{\nu\mu}{}_\lambda\right)\\ \Rightarrow \partial_\nu\left(\sqrt{g}g^{\mu\nu}\right) &= -\sqrt{g}\left(\Gamma^{\mu\nu}{}_\nu - T^{\nu\mu}{}_\nu\right).\end{aligned}$$

From $\det(\exp(g)) = \exp(\mathrm{tr}(g)) \Rightarrow \ln(\det(g)) = \mathrm{tr}(\ln(g))$, we can take the derivative of the components upon which it turns out that

$$\begin{aligned}\frac{1}{\det(g)}\partial_\lambda(\det(g)) &= \mathrm{tr}\left(g^{-1}\partial_\lambda g\right)\\ &= g^{\mu\nu}\partial_\lambda g_{\mu\nu}\\ \Rightarrow \partial_\lambda\sqrt{g} &= \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\lambda g_{\mu\nu}\\ \Rightarrow \frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} &= \Gamma^\mu{}_{\lambda\mu} - T^\mu{}_{\mu\lambda}\\ &= \overline{\Gamma}^\mu{}_{\lambda\mu}.\end{aligned}$$

By considering the inverse matrix, we see that these expressions are also valid with $g^{\mu\nu}\partial_\lambda g_{\mu\nu} \to -g_{\mu\nu}\partial_\lambda g^{\mu\nu}$. The first line above may also be applied to the tensor Lie derivative in terms of coordinates, yielding

$$\begin{aligned}L_u\sqrt{g} &= \sqrt{g}\ \mathrm{div}(u)\\ \Rightarrow \mathrm{div}(u) &= \frac{1}{2}g^{\mu\nu}L_u g_{\mu\nu}.\end{aligned}$$

If we consider an anti-symmetric tensor $F^{\mu\nu}$ and a symmetric tensor $G^{\mu\nu}$, it is not hard to see that

$$\begin{aligned}\nabla_\nu F^{\mu\nu} - T^\lambda{}_{\lambda\nu}F^{\mu\nu} &= \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}F^{\mu\nu}\right) - \frac{1}{2}T^\mu{}_{\lambda\nu}F^{\lambda\nu},\\ \nabla_\nu G^{\mu\nu} - T^\lambda{}_{\lambda\nu}G^{\mu\nu} &= \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}G^{\mu\nu}\right) + \Gamma^\mu{}_{\lambda\nu}G^{\lambda\nu},\\ \nabla_\nu G_\mu{}^\nu - T^\lambda{}_{\lambda\nu}G_\mu{}^\nu &= \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}G_\mu{}^\nu\right) - \Gamma^\lambda{}_{\mu\nu}G_\lambda{}^\nu\\ &= \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}G_\mu{}^\nu\right) - \frac{1}{2}\partial_\mu g_{\lambda\nu}G^{\lambda\nu} + T^\lambda{}_{\mu\nu}G_\lambda{}^\nu.\end{aligned}$$

The above expressions are more commonly presented with zero torsion, with $\overline{\nabla}_\nu$ defining the "divergence" of the tensor. It can also be shown (Frankel (1997) p. 365) that the "divergence" of an exterior k-form expressed as an anti-symmetric tensor can be written in terms of the hodge star as

$$\begin{aligned}\overline{\nabla}^\nu F_{\nu\mu_2\cdots\mu_k} &\equiv g^{\nu\mu_1}\overline{\nabla}_\nu F_{\mu_1\cdots\mu_k}\\ &= -(\delta F)_{\mu_2\cdots\mu_k}\\ &= (-1)^{n(k+1)+s}\left(*\mathrm{d}(*F)\right)_{\mu_2\cdots\mu_k}.\end{aligned}$$

Coordinate and tensor divergence theorems

p202, 9.3.4 The divergence and conserved quantities: The expression for the divergence theorem is

$$\begin{aligned}\int_V \mathrm{div}(u)\mathrm{d}V &= \int_{\partial V} i_u \mathrm{d}V\\ &= \int_{\partial V}\langle u, \hat{n}\rangle\,\mathrm{d}S,\end{aligned}$$

where V is an n-dimensional compact submanifold of M^n, $\hat{n}$ is the unit normal vector to ∂V, and $\mathrm{d}S \equiv i_{\hat{n}}\mathrm{d}V$ is the induced volume element ("surface element") for ∂V. If we choose an orthonormal frame with $e_1 = \hat{n}$ on ∂V, the divergence theorem can be written

$$\int_V \mathrm{div}(u)\mathrm{d}V = \int_{\partial V} u^1 \mathrm{d}S,$$

and if we can choose coordinates with x^1 constant on ∂V and normal to it, the divergence theorem can be written

$$\begin{aligned}\int_V \partial_\lambda \left(\sqrt{g} u^\lambda\right) \mathrm{d}^n x &= \int_{\partial V} \sqrt{g} \mathrm{d}x^1 (u)\, \mathrm{d}^{n-1} x \\ &= \int_{\partial V} u^1 \sqrt{g} \mathrm{d}^{n-1} x,\end{aligned}$$

where $\mathrm{d}^n x \equiv \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $\mathrm{d}^{n-1} x \equiv \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

Since the "divergence" of a tensor T with order greater than 1 is tensor-valued, and the parallel transport of tensors is path-dependent, we cannot in general integrate to get a divergence theorem for tensors. In the case of a flat metric and zero torsion however, we can choose coordinates whose coordinate frame is orthonormal, so that the frame is its own parallel transport, i.e. $\nabla_v (\beta^\mu) = 0$. For e.g. a tensor T^{ab}, we can then define a coordinate-dependent vector J^μ for each index μ

$$\begin{aligned} J^\mu &\equiv T(\beta^\mu, \quad) \\ \Rightarrow (J^\mu)^b &= T^{\mu b} \\ \Rightarrow \overline{\nabla}_v J^\mu &\overset{\not{R}}{=} \beta^\mu \overline{\nabla}_v T \\ \Rightarrow \overline{\nabla}_b (J^\mu)^b &\overset{\not{R}}{=} \overline{\nabla}_b T^{\mu b} \\ \Rightarrow \int_V \overline{\nabla}_b T^{\mu b} \mathrm{d}V &\overset{\not{R}}{=} \int_V \overline{\nabla}_b (J^\mu)^b \, \mathrm{d}V \\ &= \int_{\partial V} T^\mu{}_b \hat{n}^b \mathrm{d}S. \end{aligned}$$

For arbitrary coordinates, the components of the coordinate frame are by definition constant, i.e. $\partial_v (\mathrm{d}x^\mu) = 0$; we can therefore write

$$\begin{aligned} \sqrt{g} J^\mu &\equiv \sqrt{g} T(\mathrm{d}x^\mu, \quad) \\ \Rightarrow \partial_\nu (\sqrt{g} J^\mu)^\nu &= \partial_\nu (\sqrt{g} T^{\mu\nu}) \\ \Rightarrow \int_V \partial_\nu (\sqrt{g} T^{\mu\nu}) \mathrm{d}^n x &= \int_V \partial_\nu (\sqrt{g} J^\mu)^\nu \mathrm{d}^n x \\ &= \int_V \nabla_b (J^\mu)^b \mathrm{d}V \\ &= \int_{\partial V} T^\mu{}_b \hat{n}^b \mathrm{d}S. \end{aligned}$$

This relation remains true in the presence of both curvature and torsion, however it is important to note that $\partial_\nu \left(\sqrt{g} T^{\mu\nu}\right)$ is not a "divergence" and $T^{\mu b} = (J^\mu)^b$ is coordinate-dependent. In the special case of an anti-symmetric tensor under zero torsion, we can write

$$\begin{aligned} \int_V \overline{\nabla}_\nu F^{\mu\nu} \mathrm{d}V &= \int_V \partial_\nu (\sqrt{g} F^{\mu\nu}) \mathrm{d}^n x \\ &= \int_{\partial V} F^\mu{}_b \hat{n}^b \mathrm{d}S. \end{aligned}$$

Tensor densities

p202, 9.3.4 The divergence and conserved quantities: $\sqrt{g}$ itself can thus be called a scalar density.

From the expressions in the preceding sections we also get

$$
\begin{aligned}
\partial_\lambda (\mathfrak{T}) &= \sqrt{g}^W \partial_\lambda T + W \left(\Gamma^\mu{}_{\lambda\mu} - T^\mu{}_{\mu\lambda} \right) \mathfrak{T} \\
&= \sqrt{g}^W \partial_\lambda T + W \, \overline{\Gamma}^\mu{}_{\lambda\mu} \mathfrak{T} \\
&= \sqrt{g}^W \partial_\lambda T + \frac{W}{2} g^{\mu\nu} \partial_\lambda g_{\mu\nu} \mathfrak{T}, \\
L_u (\mathfrak{T}) &= \sqrt{g}^W L_u T + W \mathrm{div}\,(u)\, \mathfrak{T} \\
&= \sqrt{g}^W L_u T + \frac{W}{2} g^{\mu\nu} L_u g_{\mu\nu} \mathfrak{T}, \\
\nabla_\lambda (\mathfrak{T}) &= \sqrt{g}^W \nabla_\lambda T,
\end{aligned}
$$

where the last is due to the covariant derivative of the metric vanishing. In particular, this means that for zero torsion the divergence of a vector density is

$$
\begin{aligned}
\overline{\nabla}_\lambda \mathfrak{J}^\lambda &= \sqrt{g} \; \overline{\nabla}_\lambda J^\lambda \\
&= \sqrt{g} \; \mathrm{div}\,(J) \\
&= \partial_\lambda \mathfrak{J}^\lambda .
\end{aligned}
$$

△ A potential source of confusion is the use of the word "density" to indicate both an amount per unit area or volume and the presence of the coordinate-dependent factor $\sqrt{g}$. Also, while the current density is metric-independent, reflecting the amount per unit time crossing a unit coordinate area instead of metric area, tensor densities in general are not necessarily metric-independent.

Conserved currents and quantities

p200, 9.3.4 The divergence and conserved quantities: We previously saw that a Lorentzian conserved current $\mathrm{div}(J) = 0$ does not imply a conserved quantity in the presence of curvature. If we are willing to consider coordinate-dependent currents, at any given point we can choose Riemann normal coordinates, which allows us to recover a conserved quantity at that point in those coordinates.

In the integral form, we may also identify a coordinate-dependent conserved quantity for a Lorentzian conserved current by integrating over a space-like volume S with coordinates such that $t \equiv x^0$ is constant on S and normal to it, while x^1 is constant on ∂S and normal to it:

$$
\begin{aligned}
0 &= \int_S \sqrt{g} \mathrm{div}(J) \mathrm{d}^3 x \\
&= \int_S \partial_\mu \mathfrak{J}^\mu \mathrm{d}^3 x \\
&= \partial_t \left(\int_S \mathfrak{J}^t \mathrm{d}^3 x \right) + \int_S \partial_i \mathfrak{J}^i \mathrm{d}^3 x \\
&= \partial_t \left(\int_S \mathfrak{J}^t \mathrm{d}^3 x \right) + \int_{\partial S} \mathfrak{J}^1 \mathrm{d}^2 x
\end{aligned}
$$

Note that the coordinate-dependent factor $\sqrt{g}$ in $\mathfrak{J} = \sqrt{g} J$ cannot be absorbed into either $\mathrm{d}^3 x$ or $\mathrm{d}^2 x$ to yield a coordinate-independent quantity. More specifically, if $\mathfrak{J}$ is either also normal to S or vanishes on ∂S, we have $\partial_t \left(\int_S \mathfrak{J}^t \mathrm{d}^3 x \right) = 0$. This also holds if S is infinite and $\mathfrak{J}$ vanishes rapidly enough at spatial infinity.

<figure>

△ A conserved quantity as we have defined it is a quantity whose amount in a volume of space changes in time by the net amount that crosses the volume boundary. This concept is not valid when $\mathrm{div}(J) = 0$ in the presence of spacetime curvature, but it is important to remember that this still means that $\int_{\partial V} \langle J, \hat{n} \rangle \, \mathrm{d}S = 0$, so that the same amount of the quantity enters and exits any finite volume of spacetime; it is in this sense that the current is "conserved."

With regard to tensors, we can conclude from our divergence theorem variants that in the case of an orthonormal coordinate frame under a flat metric and the Levi-Civita covariant derivative, we have a coordinate-dependent conserved quantity for each component of a tensor, corresponding to a coordinate-dependent conserved current:

$$\begin{aligned} \overline{\nabla}_\nu T^{\mu\nu} &= 0 \\ \Rightarrow \partial_0 T^{\mu 0} &\overset{\not{R}}{=} -\overline{\nabla}_j T^{\mu j}, \\ \int_{\partial V} T^\mu{}_b \hat{n}^b \mathrm{d}S &\overset{\not{R}}{=} 0 \end{aligned}$$

In the special case of an anti-symmetric tensor and the Levi-Civita covariant derivative we also have a divergence theorem, and therefore a coordinate-dependent conserved current for each component:

$$\begin{aligned} \overline{\nabla}_\nu F^{\mu\nu} &= 0 \\ \Rightarrow \int_{\partial V} F^\mu{}_b \hat{n}^b \mathrm{d}S &= 0 \end{aligned}$$

Sectional curvature

p204, 9.3.5 Ricci and sectional curvature: The sectional curvature is *defined* to vanish for equal arguments, where otherwise it would be undefined, since it depends upon a plane.

p205, 9.3.5 Ricci and sectional curvature: It is on a Riemannian manifold that the Ricci tensor and Einstein tensor can be diagonalized. The Einstein tensor vanishes iff the Ricci tensor does for $n > 2$; such a manifold is called **Ricci-flat**. For each value of b in an orthonormal frame in the relation $\overline{\nabla}_a \overline{G}^{ab} = 0$, this relation expressed in terms of the Riemann curvature tensor can be seen to be equivalent to the second Bianchi identity. It is for a general Lorentzian metric that there is no conserved quantity which can be associated with the vanishing divergence of the Einstein tensor.

p205, 9.3.5 Ricci and sectional curvature: There are different ways to define H^n concretely, one being the region of $\mathbb{R}^n$ with $x_0 > 0$ and metric $\delta_{\mu\nu}/x_0^2$, another being the submanifold of $\mathbb{R}^{n,1}$ with $x_0 > 0$ and $\langle x, x \rangle = -1$. Note that even if defined as a submanifold of a Lorentzian manifold, H^n is a Riemannian manifold. In contrast, the Lorentzian manifold dS_n defined as the submanifold of $\mathbb{R}^{n,1}$ with $\langle x, x \rangle = 1$ is called **de Sitter space**; it has constant $K = 1$ and so is a Lorentzian analog of S^n, but within $\mathbb{R}^{n,1}$ is a hyperboloid of one sheet, i.e. it has topology $\mathbb{R} \times S^3$. The Lorentzian manifold AdS_n defined as the submanifold of $\mathbb{R}^{n-1,2}$ with $\langle x, x \rangle = -1$ is called **anti de Sitter space**; it has constant $K = -1$ and so is a Lorentzian analog of H^n, and within $\mathbb{R}^{n-1,2}$ is a hyperboloid of one sheet. However, while this again results in topology $\mathbb{R} \times S^3$, in this case the spherical component includes closed time-like curves; to avoid this, AdS_n is sometimes defined as the universal covering space, which has topology $\mathbb{R}^4$.

△ The different definitions of the **pseudosphere** are a possible source of confusion. H^2 is sometimes called the pseudosphere, since it has constant curvature like the sphere, but the curvature is negative instead of positive. However, this term more commonly refers to the tractroid, a $K = -1$ object with a different topology. In addition, dS_2 is also sometimes called the pseudosphere, since it is defined by $\langle x, x \rangle = 1$ like the sphere, but in $\mathbb{R}^{2,1}$ instead of $\mathbb{R}^3$. Thus the term pseudosphere may refer to an object with a constant sectional curvature which is either positive or negative.

p206-9, 9.3.6 Curvature and geodesics: A more specific justification of the assumption that $\check{R}(e_1, e_2)\vec{e}_2$ is parallel to e_1 is to note that if we drop it, the only impact is that of an e_3 component on the area calculation; to address this, a more accurate picture would be to extend the area to include all four quadrants defined by both negative and positive values of e_1 and e_3, in which case any change in area due to an e_3 component cancels. Also, the ratio of acceleration to initial length (and area) is better denoted $\left.\frac{\ddot{L}}{L}\right|_{t=0}$.

p210, 9.3.7 Jacobi fields and volumes: The expression $\mathrm{D}_t^2 J = -K(J, \dot{\phi})$ depends on the assumption that $\check{R}(e_1, e_2)\vec{e}_2$ is parallel to e_1; dropping this assumption yields the expression in terms of curvature.

Gauge group

p218, 10.1.1 Matter fields and gauges: The infinite-dimensional group of maps γ^{-1} under composition is sometimes called the **global gauge group**, with G or its reps then called the **local gauge group**.

Gauge potential and field strength

p218, 10.1.2 The gauge potential and field strength: The covariant derivative for a matter field is sometimes called the **gauge covariant derivative**. The expression for $\mathrm{D}_\mu \vec{\Phi}$ is not coordinate-dependent, and can therefore also be written using an abstract index. The definition $\check{\Gamma} \equiv -iq\check{A}$ is the convention with a mostly pluses metric signature; with a mostly minuses signature the sign is reversed. However, one also finds this definition in terms of an elementary charge $e \equiv \pm q$, which may be positive or negative depending on convention, again reversing the sign. The gauge potential is also called the four-potential, or four-vector potential.

p221, 10.1.3 Spinor fields: In the figure, note that the field components shown at $p+\varepsilon v$ are those of the field value at p applied to the frame at $p+\varepsilon v$, i.e. the top right field vector depicted would be more precisely written $\phi^\alpha|_p \, e_\alpha|_{p+\varepsilon v}$; in particular, this quantity is unrelated to the value of the field $\phi^\alpha|_{p+\varepsilon v}$.

Defining bundles

p223, 10.2.1 Fiber bundles: The bundle projection is a surjective submersion. If F is given additional structure, f_i must remain an isomorphism with respect to this structure. The base space also may be called a basis.

△ A **fibration** is a topological structure similar to a fiber bundle whose definition (which we omit) implies that the fibers are homotopy equivalent instead of homeomorphic. For our purposes, the important fact is that every fiber bundle over a manifold is a fibration, but a fibration may not be a fiber bundle.

The fibers of a fiber bundle (or a fibered manifold) are leaves of a foliation, but the converse is not true since the definition is only local (e.g. leaves may not be diffeomorphic, or may be submanifolds but not embedded). An example is the Möbius strip depicted above, which can be foliated by circles, but is not a circle bundle.

p225, 10.2.2 G-bundles: The structure group G is a subgroup of the group of homeomorphisms from F to itself. If each transition function is constant within its neighborhood intersection, the fiber bundle is called **locally constant**; in this case the foliations in each neighborhood with leaves U_i, defined by the right hand side of $\phi_i \colon \pi^{-1}(U_i) \to U_i \times F$, may be stitched together to form a foliation of E. Again, an example is the foliation of the Möbius strip by circles.

Gauge transformations on frame bundles

p238, 10.3.4 Gauge transformations on frame bundles: Automorphism gauge transformations are a subset of neighborhood-wise gauge transformations since these neighborhood-wise transformations are not necessarily consistent in $U_i \cap U_j$.

Horizontal equivariant forms and automorphisms

p245, 10.3.6 Vertical tangents and horizontal equivariant forms: The derivation of the transformation of $\vec{\varphi}_i$ under an automorphism gauge transformation is more explicitly written

$$\begin{aligned}
\vec{\varphi}'_i &= \sigma_i^* \left(\gamma^{-1}\right)^* \vec{\varphi}_P \\
&= \left(\gamma^{-1}\sigma_i\right)^* \vec{\varphi}_P \\
&= \left(\gamma_i^{-1}\sigma_i\right)^* \vec{\varphi}_P \\
&= \sigma_i^* \left(\gamma_i^{-1}\right)^* \vec{\varphi}_P \\
&= \sigma_i^* \check{\gamma}_i \vec{\varphi}_P \\
&= \check{\gamma}_i \vec{\varphi}_i,
\end{aligned}$$

where we have used $\left(g(h)\right)^* \varphi = h^* \left(g^* \varphi\right)$ twice and in the penultimate line we used the equivariance of $\vec{\varphi}_P$.

Connections on bundles

p247, 10.4.1 Connections on bundles: to be clear, the principal connection 1-form is required to preserve horizontal vectors under right translation; such a principal connection 1-form exists on any principal bundle. The horizontal lift can be defined on any smooth bundle, where the bundle parallel transporter is a diffeomorphism between fibers.

Curvature on bundles

p251, 10.4.4 Curvature on principal bundles: On a smooth bundle with connection, the exterior covariant derivative gives us a definition for the curvature of the Ehresmann connection, the horizontal vector-valued 2-form

$$\vec{R} \equiv \mathrm{D}\vec{\Gamma}.$$

The horizontal tangent spaces of $\vec{\Gamma}$ define a foliation of the bundle iff this curvature vanishes.

Characteristic classes

p262, 10.5.2 Characteristic classes: the cohomology coefficient ring is commutative and unital, and the values 0 and 1 are the ring zero and unity.

Combining bundles

p263, 10.5.3 Related constructions and facts: The tensor product of two vector bundles with the same base space $(E, M, \mathbb{K}^m)$ and $(E', M, \mathbb{K}^n)$ is another vector bundle

$$(E \otimes E', M, \mathbb{K}^{mn}).$$